

CONFORMALLY FLAT PSEUDO-RIEMANNIAN HOMOGENEOUS RICCI SOLITONS 4-SPACES

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ABSTRACT. We consider four-dimensional conformally flat homogeneous pseudo-Riemannian manifolds. According to forms (Segre types) of the Ricci operator, we provide a full classification of four-dimensional pseudo-Riemannian conformally flat homogeneous Ricci solitons.

1. INTRODUCTION

A natural generalization of an Einstein manifold is Ricci soliton, i.e. a pseudo Riemannian metric g on a smooth manifold M , such that the equation

$$(1.1) \quad \mathcal{L}_X g = \varsigma g - \varrho,$$

holds for some $\varsigma \in \mathbb{R}$ and some smooth vector field X on M , where ϱ denotes the Ricci tensor of (M, g) and \mathcal{L}_X is the usual Lie derivative. According to whether $\varsigma > 0$, $\varsigma = 0$ or $\varsigma < 0$ a Ricci soliton g is said to be shrinking, steady or expanding, respectively. A homogeneous Ricci soliton on a homogeneous space $M = G/H$ is a G -invariant metric g for which the equation (1.1) holds and an invariant Ricci soliton is a homogeneous space, such that equation (1.1) is satisfied by an invariant vector field. Indeed, the study of Ricci solitons homogeneous spaces is an interesting area of research in pseudo-Riemannian geometry. For example, evolution of homogeneous Ricci solitons under the bracket flow [21], algebraic solitons and the Alekseevskii Conjecture properties [22], conformally flat Lorentzian gradient Ricci solitons [26], properties of algebraic Ricci Solitons of three-dimensional Lorentzian Lie groups [3], algebraic Ricci solitons [2]. Non-Kähler examples of Ricci solitons are very hard to find yet (see [12]). Let (G, g) be a simply-connected completely solvable Lie group equipped with a left-invariant metric, and $(\mathfrak{g}, \langle, \rangle)$ be the corresponding metric Lie algebra. Then (G, g) is a Ricci soliton if and only if $(\mathfrak{g}, \langle, \rangle)$ is an algebraic Ricci soliton [23].

A pseudo-Riemannian manifold $\mathcal{M} = (M, g)$ is said to be homogeneous if the group G of isometries acts transitively on \mathcal{M} . In this case, (M, g) can be written as $(G/H, g)$, where H is the isotropy group at a fixed point \circ of M and g is an invariant pseudo-Riemannian metric. Homogeneous manifolds have been used in several modern research in pseudo-Riemannian geometry, for example, Lorentzian spaces for which all null geodesics are homogeneous became relevant in physics [28], [14] which following this fact, several studies on *g.o.* spaces (that is, spaces whose geodesics are all homogeneous) have done in last years (see [7], [8], [9]). Homogeneous solutions to the Ricci flow have long been studied by many authors (e.g., [15], [16], [17], [24], [29], [27], [1]).

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The complete local classification of four-dimensional homogeneous pseudo-Riemannian manifolds with non-trivial isotropy obtained [19],[20]. Using that the classification of four-dimensional conformally flat at homogeneous pseudo-Riemannian manifolds $M = G/H$ was obtained in [6].

Conformally flat spaces are the subject of many investigations in Riemannian and pseudo-Riemannian geometry. A conformally flat (locally) homogeneous Riemannian manifold is (locally) symmetric [31], and so, as proved in [30], it admits a universal covering either a spaces form \mathbb{R}^n , $\mathbb{S}^n(k)$, $\mathbb{H}^n(-k)$, or one of the Riemannian products $\mathbb{R} \times \mathbb{S}^{n-1}(k)$, $\mathbb{R} \times \mathbb{H}^{n-1}(-k)$ and $\mathbb{S}^p(k) \times \mathbb{H}^{n-p}(-k)$.

Riemannian locally conformally flat complete shrinking and steady gradient Ricci solitons were recently classified [10],[13]. Conformally flat Einstein pseudo-Riemannian manifolds have constant sectional curvature. By the way, they are symmetric. Conformally flat homogeneous Riemannian manifolds are always symmetric[31]. On the other hand, some of our examples show the existence of conformally flat homogeneous pseudo-Riemannian Ricci solitons which are not symmetric. The purpose of this paper is to investigate four-dimensional conformally flat homogeneous Ricci solitons pseudo-Riemannian manifolds by focusing on the Segre types of their Ricci operator. First we give a details of local classification of four-dimensional conformally flat homogeneous pseudo-Riemannian manifolds obtained in [6], then following that we study and classify four-dimensional conformally flat homogeneous Ricci solitons.

This paper is organized as follows. In Section 2, we recall some basic facts on Ricci solitons, which play important roles in studying homogeneous Ricci solitons. In Section 3, we report some necessary results on conformally flat homogeneous pseudo-Riemannian manifolds obtained in [6]. In Section 4 we shall investigate several geometric properties of four-dimensional conformally flat pseudo-Riemannian homogeneous Ricci solitons.

2. PERLIMINARIES

Let $M = G/H$ be a homogeneous manifold (with H connected), \mathfrak{g} the Lie algebra of G and \mathfrak{h} the isotropy subalgebra. Consider $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ the factor space, which identifies with a subspace of \mathfrak{g} complementary to \mathfrak{h} . The pair $(\mathfrak{g}, \mathfrak{h})$ uniquely defines the isotropy representation

$$\psi : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{m}), \quad \psi(x)(y) = [x, y]_{\mathfrak{m}},$$

for all $x \in \mathfrak{g}, y \in \mathfrak{m}$. Suppose that $\{e_1, \dots, e_r, u_1, \dots, u_n\}$ be a basis of \mathfrak{g} , where $\{e_j\}$ and $\{u_i\}$ are bases of \mathfrak{h} and \mathfrak{m} respectively, then with respect to $\{u_i\}$, H_j would be the isotropy representation for e_j . A bilinear form B on \mathfrak{m} is invariant if and only if $\psi(x)^t \circ B + B \circ \psi(x) = 0$, for all $x \in \mathfrak{h}$, where $\psi(x)^t$ denotes the transpose of $\psi(x)$. In particular, requiring that $B = g$ is symmetric and nondegenerate, this leads to the classification of all invariant pseudo-Riemannian metrics on G/H .

Following the notation given in [6], g on \mathfrak{m} uniquely defines its invariant linear Levi-Civita connection, as the corresponding homomorphism of \mathfrak{h} -modules $\Lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda(x)(y_{\mathfrak{m}}) = [x, y]_{\mathfrak{m}}$ for all $x \in \mathfrak{h}, y \in \mathfrak{g}$. In other word

$$(2.2) \quad \Lambda(x)(y_{\mathfrak{m}}) = \frac{1}{2}[x, y]_{\mathfrak{m}} + v(x, y),$$

for all $x, y \in \mathfrak{g}$, where $v : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{m}$ is the \mathfrak{h} -invariant symmetric mapping uniquely determined by

$$2g(v(x, y), z_{\mathfrak{m}}) = g(x_{\mathfrak{m}}, [z, y]_{\mathfrak{m}}) + g(y_{\mathfrak{m}}, [z, x]_{\mathfrak{m}}),$$

for all $x, y, z \in \mathfrak{g}$. Then the curvature tensor can be determined by

$$(2.3) \quad R : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{gl}(\mathfrak{m}), \quad R(x, y) = [\Lambda(x), \Lambda(y)] - \Lambda([x, y]),$$

and with respect to u_i , the Ricci tensor ϱ of g is given by

$$(2.4) \quad \varrho(u_i, u_j) = \sum_{k=1}^4 g(R(u_k, u_i)u_j, u_k), \quad i, j = 1, \dots, 4.$$

Furthermore, whenever $X = \sum_{k=1}^4 x_k e_k$ the Equation (1.1) becomes

$$\sum_{k=1}^4 x_k (g([u_k, u_i], u_j) + g(u_i, [u_k, u_j])) + \varrho(u_i, u_j) = \varsigma g_{ij}, \quad i, j = 1, \dots, 4.$$

Moreover, the Equation (2.2) characterizes conformally flat pseudo-Riemannian manifolds of dimension $n \geq 4$, while it is trivially satisfied by any three-dimensional manifold. Conversely, the condition

$$(2.5) \quad \nabla_i \varrho_{jk} - \nabla_j \varrho_{ik} = \frac{1}{2(n-2)} (g_{jk} \nabla_i \tau - g_{ik} \nabla_j \tau),$$

which characterizes three-dimensional conformally flat spaces, is trivially satisfied by any conformally flat Riemannian manifold of dimension greater than three.

Following the notation and the classification used in [20], the space identified by the type $n.m^k : q$ is the one corresponding to the q -th pair $(\mathfrak{g}, \mathfrak{h})$ of type $n.m^k$, where $n = \dim(\mathfrak{h})$ ($= 1, \dots, 6$), m is the number of the complex subalgebra $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{so}(4, \mathbb{C})$ and k is the number of the real form of $\mathfrak{h}^{\mathbb{C}}$. When the index q is removed, we refer simultaneously to all homogeneous spaces corresponding to pairs $(\mathfrak{g}, \mathfrak{h})$ of type $n.m^k$.

We now recall the possible Segre types of the Ricci operator for a conformally flat homogeneous four-dimensional manifold through the following tables [6].

Table 1: Segre types of Q for an inner product of signature $(2, 2)$.

Case	Ia	Ib	Ic	IIa	IIb
Non degenerate type	—	$[1, 11\bar{1}]$	$[1, \bar{1}11]$	—	$[2\bar{2}]$
Degenerate type	$[(11), (11)]$ $[(1 1, 1 1)]$ $[(11, 1)1]$ $[1(1, 11)]$ $[(11, 11)]$	$[(1, 1)11]$	$[(1, 111)]$	$[(1, 1)2]$ $[1, (12)]$ $[(1, 12)]$	$[(22)]$
Case	IIId	IIc	IIIa	IIIb	IV
Non degenerate type	$[21\bar{1}]$	$[2\bar{2}]$	$[13]$	$[1, 3]$	$[4]$
Degenerate types	—	—	$[(13)]$	$[(1, 3)]$	—

Table 2: Segre types of Q for a Lorentzian inner product.

Case	Ia	Ib	II	III
Non degenerate type	—	$[11, 1\bar{1}]$	—	$[1, 3]$
Degenerate type	$[(11), (1, 1)]$ $[1(11, 1)]$ $[(111)1]$ $[(111, 1)]$	$[(11), 1\bar{1}]$	$[(11), 2]$ $[1(1, 2)]$ $[(11, 2)]$	$[(1, 3)]$

Theorem 2.1. [18] *Let M_q^n be an $n(\geq 3)$ -dimensional conformally flat homogeneous pseudo-Riemannian manifold with diagonalizable Ricci operator. Then, M_q^n is locally isometric to one of the following:*

- (i) *A pseudo-Riemannian space form;*
- (ii) *A product manifold of an m -dimensional space form of constant curvature $k \neq 0$ and an $(n - m)$ -dimensional pseudo-Riemannian manifold of constant curvature $-k$, where $2 \leq m \leq n - 2$;*
- (iii) *A product manifold of an $(n-1)$ -dimensional pseudo-Riemannian manifold of index $q - 1$ of constant curvature $k \neq 0$ and a one-dimensional Lorentzian manifold, or a product of an $(n - 1)$ -dimensional pseudo-Riemannian manifold of index q of constant curvature $k \neq 0$ and a one-dimensional Riemannian manifold.*

It is obvious from the last theorem that if (M, g) have digonalizable Ricci operator then the Ricci operator is degenerate. So the study of cases with nondegenerate Ricci operator restricts to the non-diagonalizable ones.

3. CASES WITH NONDEGENERATE RICCI OPERATOR

As we mentioned befor if (M, g) have digonalizable Ricci operator then the Ricci operator is degenerate [6]. So let (M, g) be a conformally flat homogeneous four dimensional manifold with nondegenerate Ricci operator. For any point $p \in M$, we have that $g(0, p) = \{0\}$ if and only if Q_p is nondegenerate. Therefore, (M, g) is locally isometric to a Lie group equipped with a left-invariant pseudo-Riemannian metric and the Ricci operator of conformally flat homogeneous pseudo-Riemannian four-manifolds can only be of Segre type $[1, 11\bar{1}]$ if g is neutral, or $[11, 1\bar{1}]$ if g is Lorentzian [6]. We report here the Lie group structure of the mentioned types and their Ricci tensor as follow.

Theorem 3.1. [6] *Let (M, g) be a conformally flat homogeneous four-dimensional manifold with the Ricci operator of Segre type $[1, 11\bar{1}]$. Then, (M, g) is locally isometric to one of the unsolvable Lie groups $SU(2) \times \mathbb{R}$ or $SL(2, \mathbb{R}) \times \mathbb{R}$, equipped with a left-invariant neutral metric, admitting a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for their Lie algebra, such that the Lie brackets take one of the following forms:*

- i) $[e_1, e_2] = \varepsilon\alpha e_3, \quad [e_1, e_3] = -\varepsilon\alpha e_2, \quad [e_2, e_3] = 2\alpha(e_1 + \varepsilon e_4),$
 $[e_2, e_4] = -\alpha e_3, \quad [e_3, e_4] = \alpha e_2,$
- ii) $[e_1, e_2] = -\varepsilon\alpha e_1, \quad [e_1, e_3] = \alpha e_1, \quad [e_1, e_4] = 2\alpha(\varepsilon e_2 - e_3),$
 $[e_2, e_4] = -\varepsilon\alpha e_4, \quad [e_3, e_4] = \alpha e_4,$

and the Ricci tensor in case (i) is given by

$$\varrho = \begin{pmatrix} -2\alpha^2 + 2\epsilon^2\alpha^2 & 0 & 0 & 4\epsilon\alpha^2 \\ 0 & 2\alpha^2 + 4\epsilon\alpha^2 - 2\epsilon^2\alpha^2 & 0 & 0 \\ 0 & 0 & 4\epsilon\alpha^2 - 2\alpha^2 + 2\epsilon^2\alpha^2 & 0 \\ 4\epsilon\alpha^2 & 0 & 0 & 2\alpha^2 - 2\epsilon^2\alpha^2 \end{pmatrix},$$

also the Ricci tensor in case (ii) is then given by

$$(3.6) \quad \varrho = \begin{pmatrix} 2\epsilon^2\alpha^2 - 2\alpha^2 & 0 & 0 & -2\epsilon^2\alpha^2 - 2\alpha^2 \\ 0 & -4\epsilon^2\alpha^2 & 0 & 0 \\ 0 & 0 & -4\alpha^2 & 0 \\ -2\epsilon^2\alpha^2 - 2\alpha^2 & 0 & 0 & -2\epsilon^2\alpha^2 + 2\alpha^2 \end{pmatrix},$$

where $\alpha \neq 0$ is a real constant and $\epsilon = \pm 1$.

Theorem 3.2. [6] *Let (M, g) be a conformally flat homogeneous Lorentzian four-manifold with the Ricci operator of Segre type $[11, 1\bar{1}]$. Then, (M, g) is locally isometric to one of the unsolvable Lie groups $SU(2) \times \mathbb{R}$ or $SL(2, \mathbb{R}) \times \mathbb{R}$, equipped with a left invariant Lorentzian metric, admitting a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the Lie algebra, such that the Lie brackets take one of the following forms:*

$$\begin{aligned} \text{i) } & [e_1, e_2] = -2\alpha(\epsilon e_3 + e_4), \quad [e_1, e_3] = \epsilon\alpha e_2, \quad [e_1, e_4] = \alpha e_2, \\ & [e_2, e_3] = \epsilon\alpha e_1, \quad [e_2, e_4] = \alpha e_1, \\ \text{ii) } & [e_1, e_2] = 2\alpha(\epsilon e_3 + e_4), \quad [e_1, e_3] = \epsilon\alpha e_2, \quad [e_1, e_4] = \alpha e_2, \\ & [e_2, e_3] = \epsilon\alpha e_1, \quad [e_2, e_4] = \alpha e_1, \end{aligned}$$

and the Ricci tensor in case (i) is given by

$$\varrho = \begin{pmatrix} 4\alpha^2 & 0 & 0 & 0 \\ 0 & -4\epsilon^2\alpha^2 & 0 & 0 \\ 0 & 0 & 0 & -4\alpha^2\epsilon \\ 0 & 0 & -4\alpha^2\epsilon & 0 \end{pmatrix},$$

also the Ricci tensor in case (ii) is then given by

$$\varrho = \begin{pmatrix} -4\alpha^2\epsilon^2 & 0 & 0 & 0 \\ 0 & 4\alpha^2 & 0 & 0 \\ 0 & 0 & 0 & -4\alpha^2\epsilon \\ 0 & 0 & -4\alpha^2\epsilon & 0 \end{pmatrix},$$

where $\alpha \neq 0$ is a real constant and $\epsilon = \pm 1$.

Now using the above classification statements we classify conformally flat homogeneous Ricci soliton four dimensional manifolds with nondegenerate Ricci operator. The result is the following theorem.

Theorem 3.3. *Let (M, g) be a conformally flat homogeneous four dimensional manifold with nondegenerate Ricci operator. Then (M, g) can not be a Ricci soliton manifold.*

Proof. According to the above argument for signature (2,2) and Lorentzian signature, we have the explicit description of Lie groups and their Lie algebras. We report the calculations for the case (ii) of signature (2,2) with $\varepsilon = \pm 1$. Using (2.2) to compute $\Lambda_i := \Lambda(e_i)$ for all indices $i = 1, \dots, 4$, so describing the Levi-Civita connection of g . We get

$$\begin{aligned}\Lambda_1 &= \begin{pmatrix} 0 & -\epsilon\alpha & \alpha & 0 \\ \epsilon\alpha & 0 & 0 & \epsilon\alpha \\ \alpha & 0 & 0 & -\alpha \\ 0 & \epsilon\alpha & \alpha & 0 \end{pmatrix}, & \Lambda_2 &= \begin{pmatrix} 0 & 0 & 0 & \epsilon\alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \epsilon\alpha & 0 & 0 & 0 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \end{pmatrix}, & \Lambda_4 &= \begin{pmatrix} 0 & \epsilon\alpha & \alpha & 0 \\ -\epsilon\alpha & 0 & 0 & \epsilon\alpha \\ \alpha & 0 & 0 & \alpha \\ 0 & \epsilon\alpha & -\alpha & 0 \end{pmatrix}.\end{aligned}$$

Ricci tensors can be now deduced from the above formulas by a direct calculation applying (2.3). In particular, the Ricci tensor in this case has the form described in (3.6) where $\alpha \neq 0$ is a real constant and $\varepsilon = \pm 1$.

We prove the existence of a vector field which determine a Ricci soliton in any possible case leads to a contradiction. Choose the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$, and an arbitrary vector field $X = \sum_{k=1}^4 x_k e_k$ and a real constant ς , by (1.1) we find that X and ς determine a Ricci soliton if and only if the components x_k of X with respect to $\{e_k\}$ and ς satisfy

$$\begin{cases} -4\alpha^2 + \varsigma = 0 \\ -4\epsilon^2\alpha^2 - \varsigma = 0 \\ -2\epsilon^2\alpha^2 - 2\alpha^2 = 0 \\ x_1\alpha - 2x_4\alpha = 0 \\ 2x_1\alpha + x_4\alpha = 0 \\ -x_1\epsilon\alpha - 2x_4\epsilon\alpha = 0 \\ 2x_1\epsilon\alpha - x_4\epsilon\alpha = 0 \\ 2x_2\epsilon\alpha - 2x_3\alpha - 2\epsilon^2\alpha^2 + 2\alpha^2 + \varsigma = 0 \\ 2x_2\epsilon\alpha - 2x_3\alpha + 2\epsilon^2\alpha^2 - 2\alpha^2 - \varsigma = 0 \end{cases}$$

From third equation we find that $\alpha = 0$ which is impossible. \square

4. CASES WITH DEGENERATE RICCI OPERATOR AND TRIVIAL ISOTROPY

By the arguments of the previous section, now we proceed the manifolds with degenerate Ricci operator. For Ricci parallel examples by Proposition 4.1 of [6] it must be noted that the conformally flat Ricci parallel homogeneous Walker spaces are one of the spaces of the Theorem 2.1, or admit a two step nilpotent Ricci operator. Now, let (M, g) be a not Ricci parallel (and so not locally symmetric) conformally flat homogeneous manifold with degenerate Ricci operator. First, we proceed the cases with trivial isotropy. Separating the diagonalizable Ricci operator cases, such spaces are locally isometric to a Lie group G , equipped with a left-invariant neutral metric, and Q has one of the Segre types: $[1, (12)]$,

$[1, 12]$, $[22]$, $[(13)]$ and $[(1, 3)]$. Also, for the Lorentzian signature, Q admits the Segre types either $[(11, 2)]$, or $[(1, 3)]$ (see [6]).

Theorem 4.1. *Let (M, g) be a conformally flat not Ricci-parallel four-dimensional Lie group with the Ricci operator of Segre types $[1, (12)]$, $[22]$, $[(13)]$ and $[(1, 3)]$, then (M, g) is not a Ricci soliton manifold.*

Proof. We apply the same argument used to prove Theorem 3.1, proving that in the case $[(22)]$ there is not any Ricci soliton. Consider the pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$, such that

$$\begin{aligned} [e_1, e_2] &= \frac{1-4k_1^2}{4k_1}e_2 + \frac{1}{8k_1}e_4, & [e_1, e_3] &= \frac{1+8k_1^2}{4k_1}e_1 + \frac{1+8k_1^2}{4k_1}e_3, \\ [e_1, e_4] &= \frac{1+16k_1^2}{8k_1}e_2 + \frac{1+4k_1^2}{4k_1}e_4, & [e_2, e_3] &= \frac{1+4k_1^2}{4k_1}e_2 + \frac{1+16k_1^2}{8k_1}e_4, \\ [e_3, e_4] &= \frac{-1}{8k_1}e_2 - \frac{1-4k_1^2}{4k_1}e_4, \end{aligned}$$

for any real constant k_1 . Using (2.2) to compute $\Lambda_i := \Lambda(e_i)$ for all indices $i = 1, \dots, 4$, so describing the Levi-Civita connection of g . We get

$$\begin{aligned} \Lambda_1 &= \begin{pmatrix} 0 & 0 & \frac{1+8k_1^2}{4k_1} & 0 \\ 0 & 0 & 0 & \frac{1+8k_1^2}{8k_1} \\ \frac{1+8k_1^2}{4k_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1+8k_1^2}{8k_1} & 0 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} 0 & -\frac{1+4k_1^2}{4k_1} & 0 & k_1 \\ \frac{-1+4k_1^2}{4k_1} & 0 & \frac{1+4k_1^2}{4k_1} & 0 \\ 0 & \frac{1+4k_1^2}{4k_1} & 0 & -k_1 \\ k_1 & 0 & k_1 & 0 \end{pmatrix}, \\ \Lambda_3 &= \begin{pmatrix} 0 & 0 & -\frac{1+8k_1^2}{4k_1} & 0 \\ 0 & 0 & 0 & -\frac{1+8k_1^2}{8k_1} \\ -\frac{1+8k_1^2}{4k_1} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1+8k_1^2}{8k_1} & 0 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} 0 & k_1 & 0 & -\frac{1+4k_1^2}{4k_1} \\ -k_1 & 0 & -k_1 & 0 \\ 0 & -k_1 & 0 & \frac{-1+4k_1^2}{4k_1} \\ -\frac{1+4k_1^2}{4k_1} & 0 & \frac{-1+4k_1^2}{4k_1} & 0 \end{pmatrix}. \end{aligned}$$

By a direct calculation the curvature and Ricci tensors can be now obtained from formulas (2.3) and (2.4). In particular, in this case the Ricci tensor has the form

$$\varrho = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

For an arbitrary vector field $X = \sum_{k=1}^4 x_k e_k$ and a real constant ς , by (1.1) we find that X and ς determine a Ricci soliton if and only if the components x_k of X with respect to

$\{e_k\}$ and ς satisfy

$$(4.7) \quad \frac{x_2}{8k_1} + \frac{x_4(1+4k_1^2)}{4k_1} = 0$$

$$(4.8) \quad -\frac{x_4(1+16k_1^2)}{8k_1} + \frac{x_2(-1+4k_1^2)}{4k_1} = 0$$

$$(4.9) \quad \frac{x_4}{8k_1} + \frac{x_2(1+4k_1^2)}{4k_1} = 0$$

$$(4.10) \quad -\frac{x_2(1+16k_1^2)}{8k_1} + \frac{x_4(-1+4k_1^2)}{4k_1} = 0$$

$$(4.11) \quad 1 + \varsigma - \frac{x_1(1+8k_1^2)}{2k_1} = 0$$

$$(4.12) \quad \frac{x_1(1+8k_1^2)}{4k_1} + \frac{x_3(1+8k_1^2)}{4k_1} = 1$$

$$(4.13) \quad -\frac{x_3(1+8k_1^2)}{2k_1} + 1 - \varsigma = 0$$

$$(4.14) \quad -\frac{x_1(-1+4k_1^2)}{2k_1} + 1 - \varsigma + \frac{x_3(1+4k_1^2)}{2k_1} = 0$$

$$(4.15) \quad -\frac{x_1(1+4k_1^2)}{2k_1} + 1 + \varsigma + \frac{x_3(-1+4k_1^2)}{2k_1} = 0$$

$$(4.16) \quad x_1\left(-\frac{1}{8k_1} + \frac{(1+16k_1^2)}{8k_1}\right) + x_3\left(-\frac{1}{8k_1} + \frac{(1+16k_1^2)}{8k_1}\right) = 1$$

The equations (4.8) and (4.9) together yield that $x_2 = x_4 = 0$, also if we plus equations (4.11) and (4.13) together then we have $\frac{(x_1+x_3)(1+8k_1^2)}{2k_1} = 2$. On the other hand, adding (4.14) to (4.15) yield that $4k_1(x_1 + x_3) = 2$. Finally, from these two equations it is clear that $1 = 0$ which means that the equations system is not compatible. \square

Let (M, g) be a conformally flat not Ricci-parallel four dimensional Lie group with the Ricci operator of Segre type $[(1, 12)]$. Then, (M, g) is locally isometric to the solvable Lie group $G = \mathbb{R} \ltimes \mathbb{R}^3$, equipped with a left invariant neutral metric, admitting a pseudo-orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for the Lie algebra, such that the Lie algebra \mathfrak{g} is described by

$$\begin{aligned} [e_1, e_2] &= -[e_1, e_3] = -\frac{1}{2k_1}e_1 - k_2e_2 - k_2e_3, & [e_2, e_3] &= \frac{2k_1^2+1}{2k_1}e_2 + \frac{2k_1^2+1}{2k_1}e_3, \\ [e_2, e_4] &= -[e_3, e_4] = k_3e_2 + k_3e_3 + k_1e_4, \end{aligned}$$

for any real constants $k_1 \neq 0, k_2, k_3$.

Theorem 4.2. *Let (M, g) be a conformally flat not Ricci-parallel four-dimensional Lie group with the Ricci operator of Segre type $[1, (12)]$, then (M, g) is a Ricci soliton manifold and this case occurs when ς is arbitrary and*

$$X = \frac{k_1}{1+2k_1^2}e_2 + \frac{k_1}{1+2k_1^2}e_3.$$

Proof. We apply (2.2) to compute $\Lambda_i := \Lambda(e_i)$ for all indices $i = 1, \dots, 4$, so describing the Levi-Civita connection of g . We get

$$\begin{aligned}\Lambda_1 &= \begin{pmatrix} 0 & 0 & -\frac{1}{2k_1} & \frac{1}{2k_1} \\ \frac{1}{2k_1} & 0 & 0 & 0 \\ \frac{1}{2k_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & \Lambda_3 &= \begin{pmatrix} 0 & k_2 & -k_2 & 0 \\ -k_2 & 0 & -\frac{1+2k_1^2}{2k_1} & -k_3 \\ -k_2 & -\frac{1+2k_1^2}{2k_1} & 0 & -k_3 \\ 0 & -k_3 & k_3 & 0 \end{pmatrix}, \\ \Lambda_2 &= \begin{pmatrix} 0 & -k_2 & k_2 & 0 \\ k_2 & 0 & \frac{1+2k_1^2}{2k_1} & k_3 \\ k_2 & \frac{1+2k_1^2}{2k_1} & 0 & k_3 \\ 0 & k_3 & -k_3 & 0 \end{pmatrix}, & \Lambda_4 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -k_1 \\ 0 & 0 & 0 & -k_1 \\ 0 & -k_1 & k_1 & 0 \end{pmatrix}.\end{aligned}$$

Ricci tensors can be now deduced from the above formulas by a direct calculation applying (2.4). In particular, the Ricci tensor has the form

$$(4.17) \quad \varrho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let $X = \sum_{k=1}^4 x_k e_k$ be an arbitrary vector field and ς a real constant, using all the needed information above, we obtain that the Ricci soliton condition (1.1) is satisfied if and only if

$$(4.18) \quad -\frac{x_1}{2k_1} + x_2 k_2 - x_3 k_2 = 0$$

$$(4.19) \quad \frac{x_2}{k_1} - \frac{x_3}{k_1} - \varsigma = 0$$

$$(4.20) \quad -2x_2 k_1 + 2x_3 k_1 + \varsigma = 0$$

$$(4.21) \quad -x_2 k_3 + x_3 k_3 - x_4 k_1 = 0$$

$$(4.22) \quad -2x_1 k_2 - 2x_4 k_3 + 1 + \varsigma - \frac{x_2(1+2k_1^2)}{k_1} = 0$$

$$(4.23) \quad -2x_1 k_2 - 2x_4 k_3 + 1 - \varsigma - \frac{x_3(1+2k_1^2)}{k_1} = 0$$

$$(4.24) \quad 2x_1 k_2 + 2x_4 k_3 - 1 + \frac{x_2(1+2k_1^2)}{2k_1} + \frac{x_3(1+2k_1^2)}{2k_1} = 0$$

Since k_1 is an arbitrary real number, from equations (4.19) and (4.20) we get that $\varsigma = 0$. Also equations (4.22) and (4.23) yield that $x_2 = x_3$. Now with these results, by (4.18), $x_1 = 0$ and by (4.21) $x_4 = 0$, also again by equations (4.22) and (4.23) we get that $x_2 = x_3 = \frac{k_1}{(1+2k_1^2)}$. So, X has the form

$$\frac{k_1}{1+2k_1^2} e_2 + \frac{k_1}{1+2k_1^2} e_3.$$

Since $k_1 \neq 0$, no Einstein cases occur. \square

5. CASES WITH DEGENERATE RICCI OPERATOR AND NON-TRIVIAL ISOTROPY

In this section we consider cases conformally flat homogeneous, not locally symmetric pseudo-Riemannian four manifold with non-trivial isotropy . For these spaces, the approach is based on the classification of four dimensional homogeneous spaces with non-trivial isotropy presented by Komrakov in [20]. In [6], the authors checked case by case the Komrakov's list for conformally flat not Ricci parallel (and so not locally symmetric) examples with degenerate and not diagonalizable Ricci operator.

By using the lists which are presented in [6] for the conformally flat non-symmetric homogeneous 4-spaces with non-trivial isotropy and non-diagonalizable degenerate Ricci operator, among them, we are able to determine some different examples of homogeneous spaces $M = G/H$ for which equation (1.1) holds for some vector fields $X \in \mathfrak{m}$ and some invariant metrics which are not Einstein.

If (M, g) be a conformally flat homogeneous, not locally symmetric pseudo-Riemannian four manifold, which its Ricci operator Q is degenerate and not diagonalizable, Then the possible Segre types of Q is either [22],[1, 12] or [11, 2] (see [6]). We can now state the following classification result.

Theorem 5.1. *Among conformally flat homogeneous non-locally symmetric four-dimensional pseudo-Riemannian non-trivial Ricci soliton with the Ricci operator of Segre types [22],[1, 12] or [11, 2], the Ricci Solitons examples are listed in the following Tables 3, 4 and 5, where the checkmark means that X is invariant for all Lie algebras of that form.*

Proof. As an example, we report here the calculations for the case **1.3¹.5**. Let $M = G/H$ be a four-dimensional homogeneous space, such that the isotropy subalgebra \mathfrak{h} is determined by conditions (5.25) and the Lie algebra \mathfrak{g} has been determined before. We apply (2.2) to compute $\Lambda_i := \Lambda(e_i)$ for all indices $i = 1, \dots, 4$, so describing the Levi-Civita connection of g . We get

$$\Lambda_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2}\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{2}\lambda \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} \frac{1}{2}\lambda & 0 & \frac{c(2+\lambda^2)}{a\lambda} & \frac{c}{a} \\ 1+\lambda^2 & -\lambda & -\frac{c+c\lambda^2+b\lambda}{a} & \frac{c\lambda}{2a} \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 1+\lambda^2 & -\frac{1}{2}\lambda \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} -1 & 0 & \frac{c}{a} & -\frac{2c}{a\lambda} \\ -\frac{1}{2}\lambda & 0 & \frac{c\lambda}{2a} & -\frac{c}{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\lambda & 1 \end{pmatrix}.$$

The curvature and Ricci tensors can be now deduced from the above formulas by a direct calculation applying (2.2) and (2.3). Also the Ricci tensor has the form described in (5.26). Now Choose the pseudo-orthonormal basis u_1, u_2, u_3, u_4 , and an arbitrary vector field $X = \sum_{k=1}^4 x_k u_k \in \mathfrak{m}$ and a real constant ς , by (1.1) we find that X and ς determine a Ricci soliton if and only if the components x_k of X with respect to $\{u_k\}$ and ς satisfy

$$\left\{ \begin{array}{l} ax_3 = 0 \\ -2x_1a + \frac{2\varsigma c}{\lambda} = 0 \\ x_4a + \varsigma a = 0 \\ x_1\varsigma a - \varsigma c = 0 \\ -x_3a(1 + \lambda^2) - a\lambda x_4 = 0 \\ -a\lambda x_3 - ax_4 - \varsigma a = 0 \\ -2a(1 + \lambda^2)x_1 + 2a\lambda x_2 + \frac{1}{2}\lambda^2 - \varsigma b = -2 \end{array} \right.$$

From first equation and taking account $a \neq 0$ we find that $x_3 = 0$. Now by fifth equation x_4 must be zero and following that by third equation $\varsigma = 0$. It is clear that $x_1 = 0$, so,

$$x_2 = \frac{\lambda^2 + 4}{4a\lambda}.$$

therefore X has the form

$$X = \frac{\lambda^2 + 4}{4a\lambda} u_2.$$

Since $x_2 \neq 0$, no Einstein cases occur. Finally, again by (5.25) we see at once that $X \in \mathfrak{m}$ is invariant if and only if $X \in \text{Span}\{u_2\}$. Therefore, X is invariant and so, determines a homogeneous Ricci soliton. \square

Table 3: Non-symmetric examples with \mathbf{Q} of Segre type [(22)].

Case	Invariant metric g	X	ς	X is invariant
1.3 ¹ :5	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + \frac{2c\lambda\mu - d\lambda^2 - \mu d - 2c\lambda}{\mu(\mu-1)}\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4$	$\frac{\mu(\mu-2)}{4a}u_1 - \frac{\lambda\mu}{4a}u_2$	0	✓
1.3 ¹ :28	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + 2\frac{1}{8\varsigma}\omega_3\omega_3 + \frac{1}{8\varsigma}\omega_4\omega_4$	$x_1u_1 - \frac{\gamma}{8a}u_2 + \varsigma u_3$	$\neq 0$	$\Leftrightarrow x_1 = 0$
1.3 ¹ :29	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + 2\frac{1}{8\varsigma}\omega_3\omega_3 - \frac{1}{8\varsigma}\omega_4\omega_4$	$x_1u_1 - \frac{\gamma}{8a}u_2 + \varsigma u_3$	$\neq 0$	$\Leftrightarrow x_1 = 0$
1.3 ¹ :30	(1) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b(\lambda^2 - \lambda)\omega_3\omega_3 - (d\lambda - d)\omega_3\omega_4 + d\omega_4\omega_4$	$\frac{\lambda^2 - 1}{4a\lambda}u_2$	0	✓
	(2) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_4 + d\omega_4\omega_4$	$x_1u_1 + \frac{b}{4ad}u_2 - \frac{1}{2d}u_3$	$-\frac{1}{2d}$	$\Leftrightarrow x_1 = 0$
	(3) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b(\mu^2 + \mu)\omega_3\omega_3 - (b\mu - d\mu - d - b)\omega_3\omega_4 + d\omega_4\omega_4$	$\frac{-1 + \mu^2}{4a}(u_1 - u_2)$	0	✓

Table 4: Non-symmetric examples with Q of Segre type $[(1, 12)]$.

Case	Invariant metric g	X	ς	X is invariant
1.1 ¹ :1	$2a\omega_1\omega_3 + 2c\omega_2\omega_4 + d\omega_4\omega_4$	(1) $\frac{-a^2+c^2+2\varsigma da^2}{4a^2c}u_2 - \varsigma u_4$ (2) $\frac{\varsigma d}{2a}u_2 - \varsigma u_4$	$[-\infty, \infty]$	✓ ✓
1.1 ¹ :2	(1) $2a\omega_1\omega_3 + 2c\omega_2\omega_4 + d\omega_4\omega_4, \lambda = 0$ (2) $2a\omega_1\omega_3 + 2c\omega_2\omega_4 + d\omega_4\omega_4$	(1) $x_1u_1 + \frac{1+2\varsigma d}{4c}u_2 + \varsigma u_4$ (2) $x_1u_1 + \frac{-1+2\lambda}{4c\lambda}u_2$	$[-\infty, \infty]$ 0	$\Leftrightarrow x_1 = 0$ ✓
1.3 ¹ :5	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 - \frac{2c}{\lambda}\omega_4\omega_4, \mu = 0$	$\frac{\lambda^2+4}{4a\lambda}u_2$	0	✓
1.3 ¹ :7	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 - 2c\omega_4\omega_4$	$\frac{-1-2\lambda\mu+\mu^2+\lambda^2}{4a}u_1$	0	✓
1.3 ¹ :12	(1) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4$ (2) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4, \lambda = 0, \mu = 0$ (3) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + d\omega_4\omega_4, \lambda = 0$ (4) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4, \lambda = 1 - \mu$ (5) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4, \mu = \frac{1}{2}$ (6) $2a(-\omega_1\omega_4 + \omega_2\omega_3) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4, \lambda = 0, \mu = \frac{1}{2}$	(1) $\frac{-1-2\lambda\mu+\mu^2+\lambda^2}{4a}u_1$ (2) $\frac{-1+2\varsigma d}{4a}u_1 + x_2u_2 + \varsigma u_4$ (3) $\frac{-1+\mu^2+2\varsigma d}{4a}u_1 + x_2u_2 + \varsigma u_4$ (4) $\frac{\mu(-1+\mu^2)}{a}u_1$ (5) $\frac{-4\lambda-3+4\lambda^2}{16a}u_1$ (6) $\frac{3+8\varsigma d}{16a}u_1 + x_2u_2 - \varsigma u_4$	0 $[-\infty, \infty]$ $[-\infty, \infty]$ 0 0 0	✓ $\Leftrightarrow x_2 = 0$ $\Leftrightarrow x_2 = 0$ ✓ ✓ $\Leftrightarrow x_2 = 0$
1.3 ¹ :19	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + 2c\omega_3\omega_4 + d\omega_4\omega_4$	$\frac{1}{4a}u_1$	0	✓
1.3 ¹ :21	$2a(-\omega_1\omega_4 + \omega_2\omega_3) + 2c\omega_3\omega_4 + d\omega_4\omega_4$	$\frac{(-2+\lambda)\lambda}{4a}u_1$	0	✓
1.3 ¹ :30	(1) $2a(\omega_2\omega_3 - \omega_1\omega_4) + b\omega_3\omega_3 + b(1-\mu)\omega_3\omega_4 + d\omega_4\omega_4, \lambda = 1$ (2) $2a(\omega_2\omega_3 - \omega_1\omega_4) + b\omega_3\omega_3 + b(1-\mu)\omega_3\omega_4 - \frac{1}{2\varsigma}\omega_4\omega_4, \lambda = 1, \mu = 0$ (3) $2a(\omega_2\omega_3 - \omega_1\omega_4) - \frac{1}{2\varsigma}\omega_3\omega_3 + b(1-\mu)\omega_3\omega_4 + d\omega_4\omega_4, \lambda = 0, \mu = 1$ (4) $2a(\omega_2\omega_3 - \omega_1\omega_4) + b\omega_3\omega_3 + b(1-\mu)\omega_3\omega_4 + d\omega_4\omega_4, \mu = 1$	(1) $\frac{-\mu^2+1}{4a\mu}u_1$ (2) $x_1u_1 + \frac{b}{4ad}u_2 + \varsigma u_3$ (3) $\frac{\varsigma d}{2a}u_1 + x_2u_2 + \varsigma u_4$ (4) $\frac{\lambda^2-1}{4a\lambda}u_2$	0 $[-\infty, \infty]$ $[-\infty, \infty]$ 0	✓ $\Leftrightarrow x_1 = 0$ $\Leftrightarrow x_2 = 0$ ✓
1.4 ¹ :10	$a(-2\omega_1\omega_3 + \omega_2\omega_2) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4, ad < 0, r = p^2 + p$	$\frac{p(p+1)}{a}u_1$	0	✓
2.2 ¹ :2	$2a(\omega_1\omega_3 + \omega_2\omega_4) + b\omega_2\omega_2$	$\frac{p^2-4}{4ap}u_4$	0	✓
2.2 ¹ :3	$2a(\omega_1\omega_3 + \omega_2\omega_4) + b\omega_2\omega_2$	$x_1u_1 + \varsigma u_2 + \frac{-1+2\varsigma b}{4a}u_4$	$[-\infty, \infty]$	$\Leftrightarrow x_1 = 0$
2.5 ¹ :4	$2a(\omega_1\omega_3 + \omega_2\omega_4) + b\omega_3\omega_3$	$\frac{2h-h^2+4P}{4a}u_1$	0	✓
3.3 ¹ :1	$2a(\omega_1\omega_3 + \omega_2\omega_4) + b\omega_3\omega_3$	$\frac{p}{a}u_1$	0	✓

Table 5: Non-symmetric examples with Q of Segre type $[(11, 2)]$.

Case	Invariant metric g	Vector field X	ς	X is invariant
1.1 ² :1	$c(\omega_1\omega_1 + \omega_3\omega_3) + 2b\omega_2\omega_4 + d\omega_4\omega_4, p = 2$	$\frac{4c^2+b^2-2\varsigma dc^2}{8c^2b}u_2 + \frac{1}{2}\varsigma u_4$	$[-\infty, \infty]$	✓
1.1 ² :2	(1) $c(\omega_1\omega_1 + \omega_3\omega_3) + 2b\omega_2\omega_4 + d\omega_4\omega_4, p = 2$	(1) $\frac{2-\varsigma d}{4b}u_2 + \frac{1}{2}\varsigma u_4$	$[-\infty, \infty]$	✓
	(2) $c(\omega_1\omega_1 + \omega_3\omega_3) + 2b\omega_2\omega_4 + d\omega_4\omega_4$	(2) $\frac{p-1}{bp}u_2 + \varsigma u_3$	0	✓
1.4 ¹ :9	$a(-2\omega_1\omega_3 + \omega_2\omega_2) + b\omega_3\omega_3 + 2c\omega_3\omega_4 - \frac{a(4r+1)}{4}\omega_4\omega_4, p = -\frac{1}{2}$	$\frac{-3+4r}{16a}u_1$	0	✓
1.4 ¹ :10	$a(-2\omega_1\omega_3 + \omega_2\omega_2) + b\omega_3\omega_3 + 2c\omega_3\omega_4 + d\omega_4\omega_4, ad > 0$	$\frac{p(1+p)}{a}u_1$	0	✓
2.5 ² :2	$2a\omega_1\omega_3 + a(\omega_2\omega_2 + \omega_4\omega_4) + b\omega_3\omega_3$	$\frac{r^2+p}{a}u_1$	0	✓
3.3 ² :1	$2a\omega_1\omega_3 + a(\omega_2\omega_2 + \omega_4\omega_4) + b\omega_3\omega_3$	$\frac{p}{a}u_1$	0	✓

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